$$dp/d\Omega_n = 0 \quad (11)$$

From Eqs. (9) and (11) it follows that a necessary condition for flutter instability is given by

$$\int \mu z_n y_n dx = 0 \quad (12)$$

For the special case of a conservative system, $z_n = y_n$ and condition Eq. (12) cannot be satisfied except for $y_n(x) \equiv 0$. This verifies the fact that a conservative system cannot exhibit flutter instability.

An alternate slope expression is given by

$$dp/d\Omega_n = P[\Omega_n - (\int z_n \mathscr{E} y_n dx / \int \mu z_n y_n dx)]^{-1} \quad (13)$$

which can be obtained from Eqs. (1) and (9). At divergence $\Omega_n = 0$ and the slope is negative, so that

$$\int z_n \mathscr{E} y_n dx / \int \mu z_n y_n dx > 0 \quad (14)$$

is a necessary condition for divergence instability.

The slopes at $P = 0$ can often yield useful information. At $P = 0$ both y_n and z_n are equal to the modes of free vibration ξ_n, which are often tabulated, and Eq. (9) becomes

$$dp/d\Omega_n = \int \mu \xi_n^2 dx / \int \xi_n \mathscr{T} \xi_n dx \quad (15)$$

If these slopes are all negative, the system may become unstable by divergence as in Fig. 1(a) and (b) or by flutter as in Fig. 1(c). However a positive slope for $n = 1$ and a negative slope for $n = 2$ would indicate probable flutter instability as depicted in Fig. 1(d).

Example

Consider a uniform cantilevered column subjected to a partial follower load at its free end.[3] The governing equation is assumed to be

$$-\mu_0 \Omega y(x) + EIy''''(x) + Py''(x) = 0 \quad (16)$$

with boundary conditions

$$y(0) = 0,\ y'(0) = 0,\ y''(l) = 0,\ EIy'''(l) + (1-\eta)Py'(l) = 0 \quad (17)$$

where the subscript n has been deleted. For $\eta = 1$ the load acts tangentially and for $\eta = 0$ it acts in a vertical direction. The adjoint system has boundary conditions

$$z(0) = 0,\ z'(0) = 0,\ EIz''(l) + \eta Pz(l) = 0,\ EIz'''(l) + Pz'(l) = 0 \quad (18)$$

and the same equation as Eq. (16).

After multiplying Eq. (16) by z and integrating with respect to x, the middle term must be integrated by parts to bring in the boundary terms which involve the load. One obtains

$$P = \left(\Omega \int_0^l \mu_0 zy\,dx - EI \int_0^l z''y''\,dx\right) \Big/ \left[\int_0^l zy''\,dx - (1-\eta)z(l)y'(l)\right] \quad (19)$$

which is stationary with respect to y and z, and this leads to the expression

$$\frac{dp}{d\Omega} = \int_0^l \mu_0 zy\,dx \Big/ \left[\int_0^l zy''\,dx - (1-\eta)z(l)y'(l)\right] \quad (20)$$

for the slopes of the loading-frequency curves. At $P = 0$ the slopes may be written as

$$\frac{dp}{d\Omega} = \int_0^l \mu_0 \xi^2\,dx \Big/ \left[\eta\xi(l)\xi'(l) - \int_0^l (\xi')^2\,dx\right] \quad (21)$$

where $\xi(x)$ denotes a free vibration mode of the column.

For $\eta = 0$ the loading is conservative and the slopes are negative for all P, giving divergence instability. For $\eta > 0.84$ the slope Eq. (21) is positive at the lowest frequency Ω_1, indicating flutter instability as in Fig. 1(d). As η increases from 0 to 0.84, a transition from divergence to flutter is therefore expected to occur. (The actual point of transition is $\eta = 0.5$, which was calculated in Ref. 3 by numerical solution of the transcendental characteristic equation.)

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Minimum Weight Passive Insulation Requirements for Hypersonic Cruise Vehicles

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WEIGHTS of thermal protection systems for hypersonic vehicles are usually determined either from steady-state heat conduction analyses or, in more detailed studies, from numerical solution of the transient heat conduction equation. Analytical solutions of the heat equation with application to passive insultation systems for hypersonic cruise vehicles are not available in the standard heat conduction text[1] or elsewhere. This Note presents such analytic solutions for two representative cases.

It is assumed that the vehicle may be idealized as consisting of an exterior structure, a layer of passive insulation, and an interior structure which may contain fuel. Because fuel is consumed during the flight, a typical point on the interior structure will be in contact with the fuel for an initial portion of the flight but not in contact for the remainder. Consequently, there are two limiting cases of interest: 1) The wet wall case—the temperature at the interior wall is held to that of the boiling point of the fuel throughout the flight. (This corresponds to the bottom of the last tank to be emptied.) 2) The dry wall case—the heat transferred through the insulation is absorbed by the interior structure and the temperature of this structure is allowed to rise accordingly. (This corresponds to the top of the first tank to be emptied.)

The insulation thicknesses for each case are given by the solution to the two heat-transfer problems associated with these two cases. Vehicle geometry and tank sequencing must then be considered in estimating the fraction of vehicle surface area over which the individual thicknesses apply in the weight calculation. Since the variables of interest (the insulation thicknesses) cannot be obtained explicitly, it is necessary to solve for the thicknesses by iteration. The thicknesses are optimized in the sense that the weight of insulation plus fuel boiloff is minimized.

The major assumptions of the analyses are as follows: 1) heat transfer is by conduction only; 2) heat transfer tangential to the exterior surface is negligible compared with normal; 3) thickness

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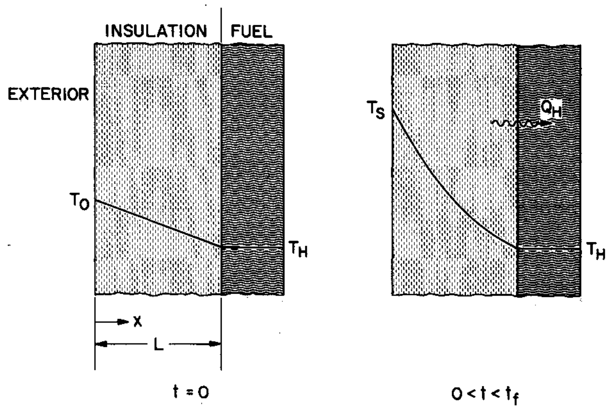

Fig. 1 Wet wall case.

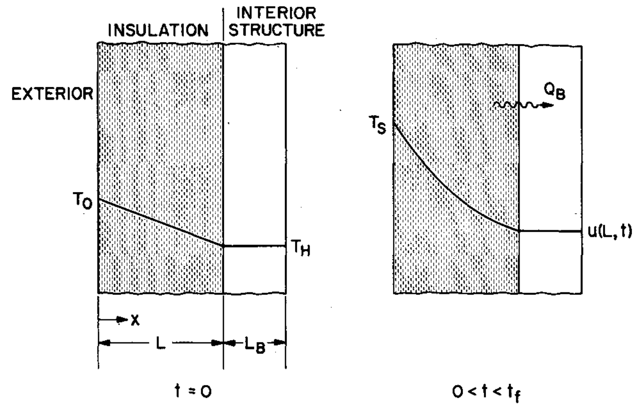

Fig. 2 Dry wall case.

of insulation is small compared with radius of exterior surface; 4) conductivity of all structural elements is infinitely large compared with conductivity of insulation; 5) insulation is continuous and homogeneous; 6) thermal constants (e.g., conductivity of insulation, specific heat of insulation, specific heat of structure) are independent of position, time, and temperature; 7) exterior is exposed to a square temperature pulse (or, equivalently, a step pulse). These assumptions imply that initial-boundary value problems of the one-dimensional heat equation are to be solved. These solutions are obtained by the standard analytical techniques of separation of variables and eigenfunction expansions (cf. Ref. 2).

Wet Wall Case

For the wet wall case, the initial-boundary value problem to be solved is

$$\partial u/\partial t = k\,\partial^2 u/\partial x^2$$
$$u(0,t) = T_S$$
$$u(L,t) = T_H$$
$$u(x,0) = [(T_H - T_0)/L]x + T_0$$

where $k = K/C\rho$ is the diffusivity of the insulation. This problem is depicted in Fig. 1. The solution of the special case $T_0 = T_H$ was obtained in Ref. 3. The solution of the general case[4] may be written

$$u(x,t) = \frac{T_H - T_S}{L}x + T_S + 2(T_0 - T_S)\sum_{n=1}^{\infty}\frac{1}{n\pi}e^{-n^2\pi^2 kt/L^2}\sin\frac{n\pi x}{L}$$

The heat transferred through the tank wall to the LH_2 fuel during a flight of time t_f is

$$Q_H = \frac{Kt_f(T_S - T_H)}{L} + \frac{2KL(T_S - T_0)}{\pi^2 k}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(1 - e^{-n^2\pi^2 kt_f/L^2})$$

Since this heat is absorbed by the fuel, the weight of insulation plus boiled-off fuel per unit of surface area is

$$W = W_I + W_{BO} = \rho L + \rho_H Q_H/h_{fg}$$

where ρ and ρ_H are the densities of the insulation and fuel, respectively, and h_{fg} is the heat of vaporization. Setting $dW/dL = 0$ for minimum W yields

$$\frac{h_{fg}\rho}{\rho_H} - \frac{Kt_f(T_S - T_H)}{L^2} - \frac{2K(T_S - T_0)}{k}\times$$
$$\left[\frac{1}{12} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n^2\pi^2} + \frac{2kt_f}{L^2}\right)e^{-n^2\pi^2 kt_f/L^2}\right] = 0$$

which is to be solved iteratively for L. After this has been done, other quantities such as Q_H and the temperature distribution $u(x,t)$ may easily be computed. It is of interest to note that for fixed materials and temperatures for optimum insulation thickness the ratio t_f/L^2 is fixed. The commonly used steady-state approximation may be readily obtained by setting $T_0 = T_S$ which gives the steady-state insulation thickness L_{SS} as

$$L_{SS} = [\rho_H K t_f (T_S - T_H)/h_{fg}\rho]^{1/2}$$

Dry Wall Case

As shown in Fig. 2, the initial-boundary value problem to be solved for the dry wall case is

$$\partial u/\partial t = k\,\partial^2 u/\partial x^2$$
$$u(0,t) = T_S$$
$$\int_0^t K\frac{u(L,\tau)}{\partial x}d\tau = -C_B L_B \rho_B[u(L,t) - T_H]$$
$$u(x,0) = [(T_H - T_0)/L]x + T_0$$

The second boundary condition is a statement that heat transferred through the insulation at $x = L$ is equal to the heat absorbed by the interior structure Q_B. Solution of the special case $T_0 = T_H$ may be obtained from the analysis of Ref. 1.

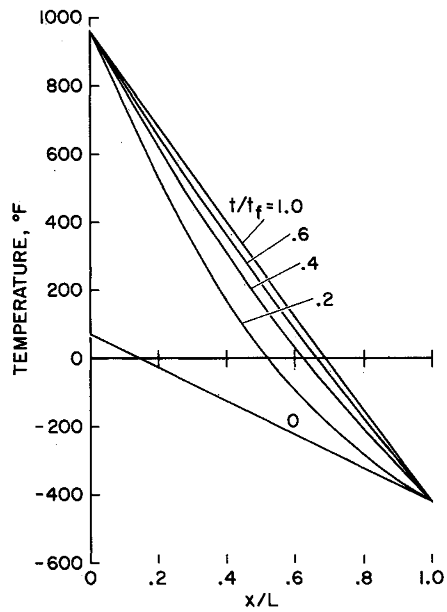

Fig. 3 Wet wall temperature distribution.

Table 1 Comparison of results of present analysis with finite difference solution and steady-state approximation

	Wet wall				Dry wall			
	L, in.	W_I, lb/ft²	W_{BO}, lb/ft²	Q_H, Btu/ft²	L, in.	W_I, lb/ft²	Q_B, Btu/ft²	T_B, °F
Present analysis	4.89	1.84	1.15	222	2.34	0.88	249	200[a]
Finite difference	4.89[a]	1.84	1.18	227	2.34[a]	0.88	248	198
Steady-state approximation	4.46	1.67	1.67	323	3.15	1.19	249	200[a]

[a] Input values.

Solution of the general problem is complicated by the fact that the eigenfunctions are not orthogonal. The solution may be written[4]

$$u(x,t) = T_S + \sum_{n=1}^{\infty} a_n e^{-\lambda_n kt} \sin[L(\lambda_n)^{1/2}]$$

It is desired to compute the insulation thickness which limits the interior surface temperature $u(L, t)$ to a specified value T_B. Since the maximum $u(L, t)$ occurs at t_f

$$T_B = T_S + \sum_{n=1}^{\infty} a_n e^{-\lambda_n kt_f} \sin[L(\lambda_n)^{1/2}]$$

which is to be iteratively solved for L. The numerical procedure for solving the dry wall problem is as follows: 1) guess an L, 2) solve for the eigenvalues λ_n from

$$-(C\rho/C_B L_B \rho_B)\cos[L(\lambda_n)^{1/2}] + (\lambda_n)^{1/2}\sin[L(\lambda_n)^{1/2}] = 0;$$
$$\pi(n-1) < L(\lambda_n)^{1/2} < \pi(n-\tfrac{1}{2})$$

3) solve for the Fourier coefficients a_n from

$$I_m = \sum_{n=1}^{\infty} a_n I_{nm}$$

where

$$I_n = (C_B L_B \rho_B/C\rho)(T_S - T_H)\sin[L(\lambda_n)^{1/2}] + (1/L\lambda_n)(T_H - T_0)\sin[L(\lambda_n)^{1/2}] + [1/(\lambda_n)^{1/2}](T_0 - T_S);\quad n = 1,2,\ldots$$

$$I_{nn} = \tfrac{1}{2}\{L - (C_B L_B \rho_B/C\rho)\sin^2[L(\lambda_n)^{1/2}]\};\quad n = 1,2,\ldots$$

$$I_{nm} = (C_B L_B \rho_B/C\rho)\sin[L(\lambda_n)^{1/2}]\sin[L(\lambda_m)^{1/2}];\quad n \neq m;\ n,m = 1,2,\ldots$$

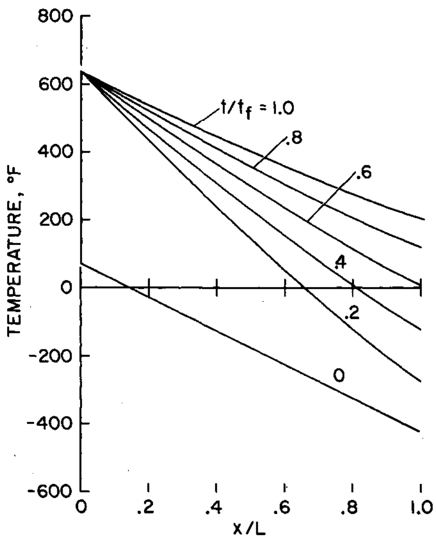

Fig. 4 Dry wall temperature distribution.

4) test to see if the equation for T_B is satisfied; if not repeat the procedure. A quasi-steady state approximation may be obtained by assuming the temperature distribution to be a linear function of x at any time t. Since $u_{SS}(L_{SS}, t_f) = T_B$ this results in

$$L_{SS} = Kt_f/\{C_B L_B \rho_B \ln[(T_S - T_H)/(T_S - T_B)]\}$$

It is instructive to consider a specific example to illustrate the solutions obtained and to provide a comparison with finite difference solutions and steady-state approximations. As compared with the present analysis, the former require substantially greater computation time while the later require substantially less. The data for the example was chosen to be representative of a liquid hydrogen fueled hypersonic cruise aircraft travelling at Mach 6 with a cruise time of about $1\frac{1}{2}$ hr.[4] The insulation is quartz fiber, and the interior structure is aluminum alloy, weighs about 3 lb/ft², and is limited to 200°F. The input temperatures were $T_0 = 70$°F, $T_{SWET} = 952$°F, $T_{SDRY} = 632$°F, and $T_H = -424$°F. The results of applying the present analysis to this example are summarized in Table 1 and the temperature distributions for the wet and dry wall cases are shown in Figs. 3 and 4, respectively.

The insulation thicknesses computed by the analytic solutions of the present paper were entered into a finite difference computer program and the resulting temperature distributions were compared with the analytically derived ones. Agreement was found to be excellent. Since there is no theoretical difference between these two solutions, this agreement indicates a high degree of numerical accuracy. The finite difference method gave a 2.0% increase in Q_H and a 0.2% decrease in Q_B as compared with the present analytic solution. The predicted final interior wall temperature differed by 2°F.

Since the steady-state approximations may be viewed as ignoring the heat capacity of the insulation, they will give conservative weights. These approximations are convenient for weight estimation because they give the weight in closed form. Table 1 shows the results of applying the steady-state approximations to the example. For the wet wall case, the steady-state approximation implies that the weight of insulation will equal the weight of boiled-off fuel for minimum total weight; whereas the transient solution indicates that the insulation weight will be greater than the boiloff weight. The steady-state approximation overestimates the insulation plus boiloff weight by 11.7% for the wet wall case and overestimates the dry wall insulation weight by 35.7%.

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